Solution of nonlinear Fokker-Planck equations

A. N. Drozdov* and M. Morillo
Universidad de Sevilla, Física Teórica, Apartado Correos 1065, Sevilla 41080, Spain
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A finite-difference method for solving a general class of linear and nonlinear time-dependent Fokker-Planck equations, which is based on a K-point Stirling interpolation formula, is suggested. It has a fifth-order convergence in time and a $2K$th-order convergence in space and allows one to achieve a given level of accuracy with a slow (or even without) increase in the number of grid points. The most appealing features of the method are perhaps that it is norm conserved, and equilibrium preserving in the sense that every equilibrium solution of the analytic equations is also an equilibrium solution of the discretized equations. The method is applied to a nonlinear stochastic mean-field model introduced by Kometani and Shimizu [J. Stat. Phys. **13**, 473 (1983)], which exhibits a phase transition. The results are compared with those obtained with other methods that rely on not too well controlled approximations. Our finite-difference scheme permits us to establish the region of validity and the limitations of those approximations. The nonlinearity of the system is found to be an obstacle for the application of Suzuki's scaling ideas, which are known to be suitable for linear problems. But what is most remarkable is that this nonlinearity allows for transient bimodality in a globally monostable case, even though there is no ''flat'' region in the potential. [S1063-651X(96)08307-9]

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I. INTRODUCTION

The Fokker-Planck equation is the basic evolution equation for a great number of physical problems. There are, however, very few models that can be solved exactly with presently known mathematical techniques. This situation gives rise to many stimulating opportunities for the development of approximate methods to analyze such problems. Computational approaches have grown enormously in popularity within this context, yielding important insights into the behavior of complex physical systems. Numerical methods of different kinds have been developed to greatly help visualization of stochastic dynamics. These methods are particularly useful for a detailed study of interesting transient phenomena, which would be difficult to approach otherwise. Most of the Fokker-Planck equations analyzed in the literature are linear in the probability density, even though the underlying Langevin dynamics might very well be nonlinear. Whereas linear Fokker-Planck equations are in principle amenable to numerical solutions, the truly nonlinear ones are much more difficult to treat and in most nontrivial cases one has to resort to *ad hoc* approximations [1,2]. Several numerical techniques exist in the literature for the analysis of nonlinear problems. Widely used methods are the cumulant expansion [1,2], path-integral techniques [3,4], and stochastic computer simulations [5]. Each of them has its own advantages when applied to the nonlinear Fokker-Planck equation. Specific advantages sought include having high, easily controllable accuracy and a rapid convergence rate, being computationally efficient in terms of speed and storage requirements, and being equilibrium preserving. The latter requirement is particularly important in studying systems that exhibit phase transitions. All the aforementioned methods, however, fail to produce exact equilibrium solutions of the Fokker-Planck operator. Common drawbacks are their hardly controllable accuracy and their too slow convergence, except perhaps in cases where the distribution function is single peaked. The difficulty arises when dealing with systems with more than one stable state, for which simulations over very long-time lengths are usually necessary.

An alternative procedure free of these drawbacks is the finite-difference method of Chang and Cooper [6] (see also its reformulations by Larsen *et al.* [7]). It is based upon the requirement that the discretized Fokker-Planck operator possesses equilibrium solutions that agree at the mesh points with equilibrium solutions of the analytic operator. The accuracy of this approach is determined by two factors: the accuracy of the time propagation scheme and the accuracy of the space discretization. The standard way to control its accuracy is thus varying the number of time integration steps and that of grid points. In practice, however, one would like these numbers to be as small as possible to achieve computational economy. From this point of view none of the methods developed in [6,7] can be recognized as suitable. These are all of first-order convergence in time and second-order convergence in space. As a result, a rather large number of grid points (about 200) is usually necessary to get results with three stable digits [7].

The purpose of this work is twofold. First, we develop a method that allows us to reach a given level of accuracy without any (or with a mild) increase in the number of space discretization points. Second, we will apply the method to numerically study some aspects of the relaxation dynamics of a nonlinear Fokker-Planck model. It was introduced by Kometani and Shimizu within the context of muscle contraction [8]. The model describes the cooperative dynamics of a large number of subunits with a mean-field interaction between them. A more complete statistical-mechanical treatment of the model given later by Desai and Zwanzig [1] and by Dawson [9] pointed out its relation with the Weiss-Ising

*Permanent address: Institute for High Temperatures, 13/19 Izhorskaya Street, 127412, Moscow, Russia.

model. These authors also showed that the combined effect of thermal noise and mean-field interaction gives rise, in the thermodynamic limit, to a truly nonlinear Fokker-Planck equation for the probability density associated with the order parameter. The system presents an order-disorder phase transition described by a bifurcation of its equilibrium distribution. The relaxation of the system from different initial conditions was studied by Desai and Zwanzig [1], as well as by Brey, Casado, and Morillo [2] by numerically solving a truncated cumulants hierarchy. One of the drawbacks of this method, however, is that it does not correspond to any systematic expansion in a parameter of the system, so it is not simple to improve its convergence. In a certain range of parameters, the cumulant expansion was found to be very slowly convergent, and Suzuki's scaling hypothesis [10] was used to overcome this problem [2]; but the range of the validity of this approximate procedure has remained unclear. So a specific goal we would like to achieve in the present work is to test the validity of these two approaches.

The outline of the article is as follows. In Sec. II a method is presented and tested on an exactly solvable Fokker-Planck equation. In Sec. III we give a brief description of the model and compare our results with those of [2]. Section IV concludes with general remarks and future applications.

II. METHOD

In this section, we outline a method for numerically solving a general class of one-dimensional Fokker-Planck equations of the form

$$\partial_t P(x,t) = L[x,t,P]P(x,t) \equiv \frac{1}{A(x)}\partial_x[B(x,t,P)+Q(x,t,P)\partial_x]P(x,t), \quad (1)$$

with the boundary and initial conditions

$$(B+Q\partial_x)P=0 \text{ at } x=x_l,\ x_r, \quad (2)$$

$$P(x,t=0)=P(x,0). \quad (3)$$

A generalization of the method to systems with more than one degrees of freedom is straightforward. Note that the Fokker-Planck operator L, defined on the right-hand side of Eq. (1), possesses quasiequilibrium solutions that are all determined by

$$[B(x,t,P)+Q(x,t,P)\partial_x]P=0. \quad (4)$$

By quasiequilibrium we mean that solutions of $L[x,t,P]P(x,t)=0$ [cf. (4)] are generally dependent of t and therefore these are not true equilibrium solutions of (1). They become equilibrium solutions only if B and Q become explicitly independent of t as t goes to infinity.

Following [7], our primary assumption is that (1) can be cast into the form

$$\partial_t P(x,t)=\frac{1}{A(x)}\partial_x D(x,t,P)E^{-2}(x,t,P)\partial_x E^2(x,t,P)P(x,t), \quad (5)$$

$$\partial_x E^2 P=0 \text{ at } x=x_l,\ x_r, \quad (6)$$

where $D(x,t,P)$ and $E(x,t,P)$ are positive and sufficiently smooth functions assumed to be known in analytical form and where all quasiequilibrium solutions are now determined by the algebraic equation

$$E^2[x,t,P(x,t)]P(x,t)=\gamma \quad (7)$$

for an appropriate set of constants γ. If the functions $D(x,t,P)$ and $E(x,t,P)$ do not possess closed analytic forms, they can be determined numerically as suitable solutions of the equations

$$B(x,t,P)\partial_P[E^2(x,t,P)P]=Q(x,t,P)\partial_x E^2(x,t,P), \quad (8)$$

$$2D(x,t,P)\partial_x E(x,t,P)=B(x,t,P)E(x,t,P), \quad (9)$$

where x and P are regarded as independent variables and t as a parameter.

In order to integrate Eq. (5) by numerically solving a finite-difference equation, one has to define a two-dimensional $t-x$ lattice [6,7]. However, for the sake of clarity, only the t variable will first be discretized, while the x variable will be left continuous. This simplification is possible because none of the following considerations depends on the way the space variable is discretized.

A straightforward integrating of (5) over time yields

$$P(t+\tau)=P(t)+\frac{\tau}{6}[LP(t+\tau)+LP(t)+4LP(t+\tau/2)] - \frac{\tau^5}{2880}LP^{(4)}(\xi), \quad (10)$$

where $t\leqslant\xi\leqslant t+\tau$ and where the following notations have been employed:

$$P(t)=P(x,t), \quad LP(t)=L[x,t,P(x,t)]P(x,t),$$

$$LP^{(n)}(t)=\partial_{t^n}^n[LP(t)].$$

Then, using a Taylor expansion of $P(t)$, including τ^4 terms, one gets

$$P(t+\tau)+P(t)=2P(t+\tau/2)+\frac{\tau^2}{2}P^{(2)}(t+\tau/2) + \frac{\tau^4}{192}P^{(4)}(\xi), \quad (11)$$

from which it follows, in a straightforward way, that

$$P(t+\tau/2)=\frac{1}{2}[P(t+\tau)+P(t)]+\frac{\tau}{8}[LP(t)-LP(t+\tau)] + O(\tau^4). \quad (12)$$

Combining Eqs. (10) and (12) provides us with an implicit time propagation scheme of fifth-order convergence in the time step τ. These equations are nonlinear with respect to $P(t+\tau)$ and must in general be solved iteratively at each time step. A procedure which works satisfactorily is to first calculate $P(t+\tau)=P(t)+\tau LP(t)$, then substitute it into the right-hand side of (10) to obtain (the second estimate for)

$P(t+\tau)$, and repeat this process until some convergence criterion is met. In practical calculations, we have found that if the time step is sufficiently small and $D(x,t,P)$ and $E(x,t,P)$ are sufficiently smooth functions of P, the above procedure converges in one or two iterations.

Proceeding further, we introduce a uniformly spaced x lattice made of N points with coordinates

$$x_i = x_l + (i-1)h, \quad i = 1, \ldots, N, \quad h = (x_r - x_l)/(N-1),$$

where h denotes the step size. The standard approach is then to approximate to second order in h the first derivatives in (5) by the difference [6,7]

$$(\partial_x F)_i = \frac{1}{h}(F_{i+1/2} - F_{i-1/2}) + O(h^2). \tag{13}$$

From the very beginning, however, it is clear that this differencing is unsuitable for achieving highly accurate results. The trick we employ is to rewrite Eq. (5) as

$$2\partial_t P = \frac{D}{E}\partial_{xx}^2 EP + \frac{1}{E}\partial_{xx}^2 DEP - DEP\partial_{xx}^2 \frac{1}{E} - EP\partial_{xx}^2 \frac{D}{E} \tag{14}$$

and then to approximate to $2K$th-order in h the second derivatives in the above equation with a central difference of the form

$$(\partial_{xx}^2 F)_i = h^{-2}\left[C_0 F_i + \sum_{j=1}^{K} C_j (F_{i+j} + F_{i-j}) \right] + O(h^{2K}), \tag{15}$$

where $F_i = F(x_i)$. The explicit expressions for the coefficients C_i are readily determined by using Stirling's interpolation formula, which is a Gaussian central difference $2K+1$ term polynomial coinciding with a given function $F(x)$ at the $2K+1$ points. This yields [11]

$$C_i = 2(-1)^{i+1} \sum_{j=1}^{K} \frac{[(j-1)!]^2}{(j-i)!(j+i)!}. \tag{16}$$

It must be pointed out that the use of central differences such as (15) is the conventional way in finite-difference theory [12] to represent derivatives of the given function $F(x)$ in terms of its values on the grid with any desired level of accuracy in h. In this representation, the operator L becomes an asymmetric $(2K+1)$-diagonal matrix whose elements are readily determined from (in the following, for simplicity, we shall not notationally distinguish the analytic operator L from its matrix representation)

$$\begin{aligned}(LP)_i = (2A_i h^2)^{-1} \sum_{j=1}^{K} C_j \{ & (D_i + D_{i+j})[\eta(N+1-i-j) \\ & \times E_{i+j} P_{i+j}/E_i - E_i P_i / E_{i+j}] + (D_i + D_{i-j}) \\ & \times [\eta(i-j) E_{i-j} P_{i-j}/E_i - E_i P_i / E_{i-j}]\},\end{aligned} \tag{17}$$

where

$$\eta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

One might expect that for fixed N the discretization procedure outlined above would produce, with increasing K, much more accurate results than obtained with (13). We have found, however, that the use of the matrix L defined by (17) with $N=31$ leads to a finite first eigenvalue of the Fokker-Planck operator varying from 10^{-5} to 10^{-7} instead of being exactly zero. One can mention at least three reasons for the poor behavior of L. First, the matrix defined by (17) does not account for the boundary conditions (6). Second, approximating the second derivative with the central difference (15) leads to inaccurate computation of certain matrix elements, particularly certain large elements in the upper left and lower right corners of the matrix. Finally, the third source of errors is attributed to the failure of the computed matrix to exactly preserve quasiequilibrium solutions of the analytic operator or, equivalently, to satisfy the condition

$$(LE^2)_i = 0 \text{ whatever } i, \tag{18}$$

which is simply a statement that E_i^{-2} is a null vector of L.

It is a simple matter to reduce the influence of the first two factors by modifying the vector P_i on which L operates. We have found, however, that a dramatic reduction of discretization error is already achieved due to a simple procedure to compute the diagonal elements so that quasiequilibrium solutions are all preserved. Specifically, we enforce (18) by the following modification of the diagonal of L:

$$\begin{aligned}(LP)_i = (2A_i h^2)^{-1} \sum_{j=1}^{K} C_j [& \eta(N+1-i-j)(D_i + D_{i+j}) \\ & \times (E_{i+j} P_{i+j}/E_i - E_i P_i / E_{i+j}) + \eta(i-j)(D_i + D_{i-j}) \\ & \times (E_{i-j} P_{i-j}/E_i - E_i P_i / E_{i-j})].\end{aligned} \tag{19}$$

It not hard to see that this modification is equivalent to replacing the original function E by the function

$$\overline{E} = \begin{cases} E & \text{if } x_l \leqslant x \leqslant x_r \\ \infty & \text{otherwise,} \end{cases}$$

for which the boundary conditions (6) are fulfilled automatically. The use of (19) has been found to lead to significantly greater accuracy. For example, for the cases we have tested we find that with this formulation machine accuracy in the computation of the first eigenvalue of the Fokker-Planck operator is achieved for all N and K.

Equations (10), (12), and (19) constitute a finite-difference scheme that reduces the original nonlinear problem to a conventional matrix-vector multiplication procedure. The advantages of the scheme are its generality, simplicity, and accuracy. It is worth noticing also its important properties, which are as follows.

(i) It leads to a band structured matrix L with $2K+1$ nonzero diagonals, thus allowing us to minimize the storage requirements and the execution time that is necessary for the matrix-vector multiplication [cf. Eq. (19)].

(ii) It has fifth-order convergence in time and $2K$th-order convergence in space. That is, with increasing K it is able to yield a desired accuracy with a mild (or even without) increase of the number of grid points N.

(iii) It automatically preserves every quasiequilibrium solution of the analytic equation.

(iv) Finally, an important property that is shared by the analytic equation and the numerical method presented is that they are both conservative. This means that multiplying Eq. (1) by $A(x)$ and integrating over x gives

$$\int_{x_l}^{x_r} dx A(x) P(x,t) = \int_{x_l}^{x_r} dx A(x) P(x,0), \tag{20}$$

which shows that the integral over all x of the product AP is a time-conserved quantity. Likewise, for the above numerical method we have

$$\sum_{i=1}^{N} A_i P_i(t) = \sum_{i=1}^{N} A_i P_i(0), \tag{21}$$

which shows the corresponding conservation property of the discretized equation. This property of the numerical scheme is particularly important in studying nonlinear Fokker-Planck equations whose solutions, in contrast to that of linear ones, do not allow for the choice of an arbitrary normalization. In other words, if $P(x,t)$ is a solution, an arbitrary constant times P is not.

We note that the present method can be used with any kind of boundary conditions different from those given by (6). To this end, it is enough to modify in an appropriate way the matrix representation (19). We note also that a discretization scheme analogous to the one given in (15) can be used if necessary to numerically solve Eqs. (8) and (9). Finally, we would like to emphasize that the underlying ideas behind the present method are easily implemented with any other pseudo spectral and/or collocation methods available in the literature. Most of them, however, lead to full matrix representations of the operator L, which are generally undesirable. For instance, for the cases we have tested we find that the accuracy obtained with a full matrix representation of L using the sinc-collocation method is comparable with that achieved in the present discretization with just $K=5$. This considerably reduces the CPU time necessary for vector-matrix multiplication, especially if N is sufficiently large, thus increasing the feasibility of time-dependent calculations.

It will now be our aim to illustrate the power of the present technique on an exactly solvable model. A benchmark model is

$$\partial_t P(x,t) = \partial_x [\omega x + \theta \langle x(t) \rangle + D \partial_x] P(x,t), \tag{22}$$

where ω, θ, and D are constants, while the moment $\langle x(t) \rangle$ is defined by

$$\langle x(t) \rangle = \int_{-\infty}^{\infty} dx P(x,t) x. \tag{23}$$

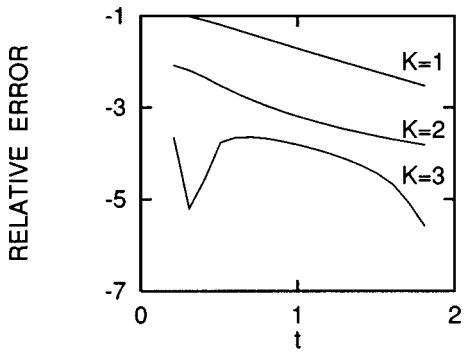

FIG. 1. Logarithm of the relative error $\log_{10}|\varepsilon(t)|$ [Eq. (28)] in the width of the propagator for a test process [Eq. (22)] for $K=1$, 2, and 3.

Notice that Eq. (22) is not an ordinary Fokker-Planck equation, as it depends on $\langle x(t) \rangle$ which is itself a functional of $P(x,t)$. It is not difficult to prove that the exact solution to (22) with the initial condition

$$P(x,t|x_0) = \delta(x - x_0) \tag{24}$$

reads

$$P(x,t|x_0) = [2\pi\sigma(t)]^{-1/2} \exp - \frac{[x - \langle x(t) \rangle]^2}{2\sigma(t)}, \tag{25}$$

$$\langle x(t) \rangle = x_0 e^{-(\omega+\theta)t}, \tag{26}$$

$$\sigma(t) = \frac{D}{\omega}(1 - e^{-2\omega t}). \tag{27}$$

Figure 1 shows the relative error made by truncating (19) at $K=1$, 2, and 3 in calculating the first two moments

$$\varepsilon(t) = \frac{\langle x^2(t) \rangle_K - \langle x(t) \rangle_K^2}{\langle x^2(t) \rangle - \langle x(t) \rangle^2} - 1, \tag{28}$$

for $D=0.1$, $\omega=1$, $\theta=1$, and $x_0=1$, where the subscript K means that the moment is not the exact one but its approximation obtained by using the present numerical scheme. It should be noted that an exponential power series expansion of the propagator [13] has been used to approximately determine $P(x,t|x_0)$ at small t's. The time step τ was taken to be the same for all K's, $\tau=0.01$. A grid of 31 points in the interval $[-1.5,1.5]$ was found to be sufficient to propagate the distribution function in time until equilibrium is reached.

It is seen that each successive higher order reduces the error over a larger range of t. It is also seen that the convergence of the solution at various orders of truncation of the central difference (15) is very rapid. For all the problems we have studied we find that quantitatively the same results are already attained by truncating at $K \geqslant 5$. As expected, beyond some $t \simeq 1/\omega$ the error begins to decrease whatever K, since the present scheme provides one with the exact equilibrium solution regardless of the order of truncation.

III. A NONLINEAR MEAN-FIELD MODEL

As a second and more challenging example, we consider a system described by the nonlinear Fokker-Planck equation

$$\partial_t P(x,t) = \partial_x [U'(x,t) + D\partial_x] P(x,t)$$
$$\equiv D\partial_x e^{-U(x,t)/D} \partial_x e^{U(x,t)/D} P(x,t),$$
$$U(x,t) = \frac{x^4}{4} + (\theta - 1)\frac{x^2}{2} - \theta \langle x(t) \rangle x, \tag{29}$$

where the prime denotes differentiation with respect to x. An important feature of this model is that it shows a genuine phase transition when the parameters D and/or θ are varied across the critical line. As follows from the analysis of the equilibrium distribution given by

$$P_e(x) = R^{-1} \exp - \frac{1}{D}\left[\frac{x^4}{4} + (\theta - 1)\frac{x^2}{2} - \theta x_e x\right], \tag{30}$$

where R is the normalization factor and

$$x_e = \int_{-\infty}^{\infty} dx P_e(x) x, \tag{31}$$

for each value of θ there exists a value of the noise strength D_c, so that for $D > D_c$ there is only one stable equilibrium distribution with $x_e = 0$, while for $D < D_c$ there are two stable equilibrium solutions with $\langle x \rangle_e = \pm x_e$, besides the zero ($\langle x \rangle_e = 0$) unstable one. Thus, at the critical line, there is a bifurcation of the equilibrium distribution function. For $D < D_c$ it is always single peaked, while for $D > D_c$ the stable equilibrium distribution has either one or two maxima depending on whether θ is larger than or less than 1. The values of D_c are determined from

$$\sqrt{2D_c} = \theta D_{-3/2}(z)/D_{-1/2}(z), \tag{32}$$

where $z = (\theta - 1)/\sqrt{2D_c}$ and $D_\nu(z)$ is a parabolic cylinder function.

A few years ago, Shiino was able to prove an H theorem for the above nonlinear Fokker-Planck equation [14]. This means that in the long-time limit, the system always reaches one of the equilibrium solutions. Clearly, for a given θ and $D > D_c$, the equilibrium is unique regardless of the initial condition. While for $D < D_c$ there are two stable equilibrium solutions and, as t goes to infinity, the system approaches one or the other depending upon the sign of $\langle x(0) \rangle$ or, in other words, upon the initial preparation of the system. In this sense, one can say that the nonlinearity breaks the ergodicity of the process.

It must be emphasized that all the above-mentioned peculiarities of the considered system were also observed in our numerical calculations, which were carried out in a wide range of parameters θ and D. We have also studied some aspects of the dynamics of this model with different initial conditions, which were previously analyzed by several workers. The main findings are as follows.

(i) As expected, the results obtained with the finite-difference method for the critical line are in excellent agreement with those obtained from (32) (see Fig. 2). This is not surprising, of course, since the method is equilibrium preserving. We show these results solely in order to illustrate the power of the present technique in yielding precise equilibrium solutions.

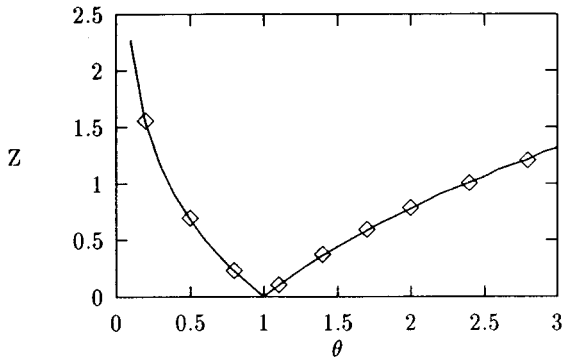

FIG. 2. Equilibrium phase diagram for the model (29). Solid line, exact results for the critical line obtained from Eq. (32); diamonds, evaluation using the present method.

(ii) The cumulant (moment) expansion works well if the system remains monostable for all t. In such a case, the convergence is rapid and neglecting cumulants beyond $n \geqslant 6$ is found to be sufficient to get accurate results in the entire time domain, except for a short initial period. In Fig. 3 we present the results for $D = 0.1$, $\theta = 2$ obtained with a δ-function initial condition (24) for $x_0 = 10^{-4}$. An exponential power-series expansion of the propagator [13] has been again used to approximately determine $P(x,t|x_0)$ at small times. It is seen that the cumulant expansion produces results that are in good agreement with those of the finite-difference method.

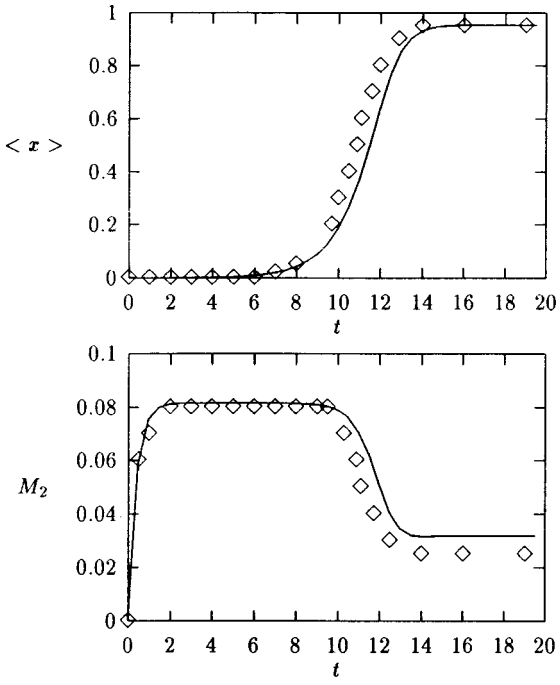

FIG. 3. Upper panel, plot of $\langle x(t) \rangle$; lower panel, $M_2(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2$ for the model (29) for $D = 0.1$, $\theta = 2$, and $x_0 = 10^{-4}$. Solid line, results obtained using the present method; diamonds, results of the sixth-order cumulant approximation.

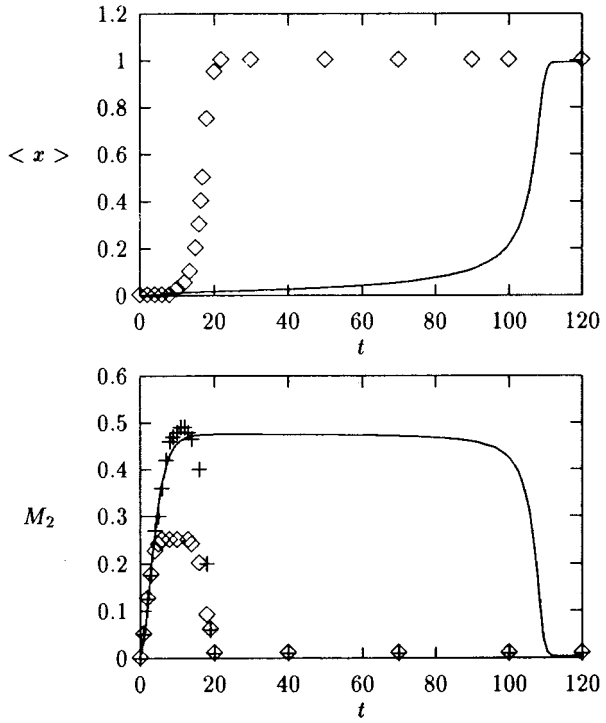

FIG. 4. Same as in Fig. 3, but for $D=0.01$ and $\theta=0.5$. The plus sign is for the results of Ref. [2].

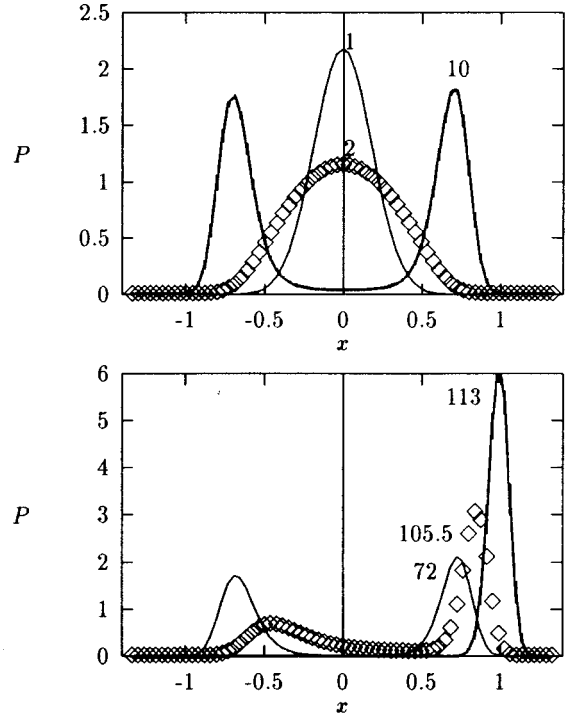

FIG. 5. Distribution function $P(x,t|x_0)$ for the model (29) for $D=0.01$, $\theta=0.5$, and $x_0=10^{-4}$ at $t=1$, 2, 10, 72, 105.5, and 113.

(iii) This is not, however, true for bistable systems ($\theta<1, D>D_c$), as well as for globally monostable systems, which demonstrates the transient bimodality for an initial period, provided that this period is not too short. Following [2], we have studied the case $\theta=0.5$ and $D=0.01$ with the same initial condition $x_0=10^{-4}$. In this case, the cumulant hierarchy was found to converge very slowly and an *ad hoc* approximation was proposed to improve its convergence. The latter rests on Suzuki's scaling idea that the Gaussian approximation is good for the initial time domain [10]. Then matching this Gaussian approximation with the sixth-order cumulant approximation, one might expect that a correct description would be attained for the whole time domain [2]. As evidenced by Fig. 4, the approximate solution thus obtained works no better than the sixth-order cumulant approximation itself. Both approximations approach equilibrium much faster than the numerically exact results do. The reason is that the Gaussian approximation appears to be applicable, in this case, for too short a period, while the nonlinearity remains important during the entire time domain. The advantage of the present finite-difference method is that it works adequately regardless of the initial condition and the number of stable states.

(iv) The calculations performed revealed immediately the existence of a sizable interval of time during which the probability distribution shows two peaks. This is clearly evident from Fig. 5, which shows the time evolution of the probability density for $\theta=0.5$, $D=0.01$, and $x_0=10^{-4}$. Transient bimodality is known to be a genuine phenomenon that can occur in linear problems. This phenomenon is usually associated with evolution in a potential having a flat plateau [15]. It arises some time after the initial probability distribution peak reaches the plateau and it disappears again once an appropriate mass of probability distribution has left the flat part of the potential. The striking difference of the present phenomenon is that the double-peaked probability distribution was observed without any long region in the potential where $U'(x,t)$ is very small. Figure 5 shows that for a very long time t the system exists with a reasonable probability in either one of two unsteady well-defined states until equilibrium is reached. Although the initial stage of relaxation is governed by an almost linear Fokker-Planck equation, $\langle x(t)\rangle \sim 10^{-4}$, the occurrence of transient bimodality is a signature of the nonlinearity of the problem.

IV. SUMMARY AND OUTLOOK

In this paper, we present a simple and easily applicable finite-difference method for numerically solving a general class of one-dimensional Fokker-Planck equations. An extension of the method to systems with more than one degree of freedom is straightforward. The method relies on the idea of writing the Fokker-Planck equation in the form (14) and on the use of a suitably modified central-difference representation of the second derivatives in x. The resulting discretized equation has two very useful properties. First, it is norm conserved and, second, all equilibrium solutions of the analytic equation, evaluated at the mesh points, are also equilibrium solutions of the discretized equation. From this point of view it is preferable to other numerical techniques available in the literature such as cumulant expansions, path-integral methods, and stochastic simulations, none of which is equilibrium preserving. It should be noted here that both properties of the present scheme become particularly important in studying truly nonlinear systems that exhibit a phase transition. In such a case, all the above-mentioned techniques may fail considerably. In particular, one must be cautious of

the use of stochastic simulation methods because their convergence, which is not so rapid by itself, becomes even slower in the vicinity of phase-transition points due to critical slowing down. As a result, one has to generate a huge number of trajectories to achieve an adequate level of accuracy, otherwise ''unexpected'' phenomena may arise. This problem is planned to be discussed in more detail in a future work [16]. It is worth noticing also the advantage of finite-difference schemes to take into account the boundary conditions. Although we restrict our considerations to the case of reflecting boundary conditions, any other boundary conditions do not seem to present special problems. The same, however, is not true for path-integral methods, where special tricks are required in order to incorporate the boundary conditions into a path integral. Finally, the present method is advantageous with respect to cumulant expansions in that it provides very accurate results irrespective of the initial condition and the number of peaks that the distribution function shows during its evolution to equilibrium.

Yet another important advantage of the present method is that it turns out to be an efficient and rather universal tool for an accurate and error-controlled treatment of the nonlinear time-dependent Fokker-Planck equation. The desired accuracy can be attained by increasing the number of nonzero diagonals $(2K+1)$ and grid points N, as well as by reducing the time step τ. The final accuracy is limited only by the requirements on the CPU time and the core of the computer being used. For the problems considered in this paper, relative precisions better than 10^{-6} are easily possible on a workstation. We believe that the method presented here will provide the necessary foundation for treating systematically and conveniently a wide variety of linear and nonlinear Fokker-Planck processes in physical, chemical, and biological systems.

The method is illustrated by applying it to a nonlinear mean-field model introduced by Kometani and Shimizu [8]. Several approximation schemes have been used so far to study the dynamics of this system in the limit of the number particles going to infinity for which a nonlinear Fokker-Planck equation was derived [1]. Usually these are based on the cumulants expansion. The advantage of this approach is that being truncated at some low order it may lead to relatively simple equations. The drawback is that it is not simple to improve the quality of the approximation thus obtained. By means of the method proposed here we are able to test the utility of this approximation itself, as well as of its *ad hoc* correction developed in [2] by combining Suzuki's scaling hypothesis with the cumulant expansion method. Our results demonstrate the validity of the cumulant expansion for essentially monostable situations. It, however, fails to produce correct results if any kind of transient and/or global bimodality occurs. It is interesting to note also that the nonlinearity involved in the system treated here turns out to be an obstacle for the application of Suzuki's scaling ideas, which are known to be suitable for linear problems. But what is most remarkable is that this nonlinearity allows for the coexistence of two peaks in the probability distribution during a long-time interval, even though there is no ''flat'' region in the potential.

Aside from the two examples that are considered in this paper, the method can also be applied to other relevant problems. For instance, it would be worthwhile to study the dynamical response of the above nonlinear model driven by a time-sinusoidal external field. In this regard, it is of particular interest to check numerically the validity of the Floquet theory within the context of nonlinear equations. Another problem of importance refers to the extension to nonstationary processes of the H theorem that is derived by Shiino [14] in the absence of external forcing. This is of interest considering the lack of a satisfactory analytical description for finite amplitudes driving fields. Both problems are investigated numerically by the present authors and the results obtained are planned to be presented elsewhere [16].

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